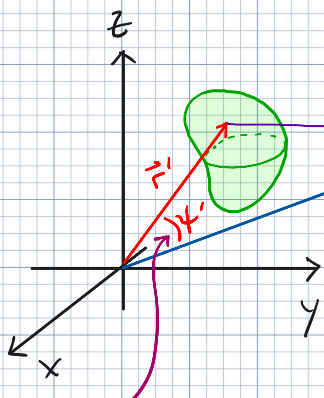
The Multipole Expansion

- A way of systematically approximating the potential produced by a distribution of charge when we're far enough away that we can ignore some of the details.



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int dq(\vec{r}\,')\,\frac{1}{\eta}$$

$$\text{w/ } \frac{1}{\eta} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi'}}$$

As we visit points w/ different $\vec{r}\,'$ & ask how charge $dq(\vec{r}\,')$ at that point contributes to V at pt. $\vec{r}$, both r' & ψ' change.

ψ' is angle b/t $\vec{r}$ & $\vec{r}\,'$:

$$\hat{r}\cdot\hat{r}' = \cos\psi'$$

- As long as $r^2 > r'^2 - 2rr'\cos\psi'$, we can write $1/\eta$ as:

$$\frac{1}{\eta} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\psi'}} = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}}\, r'^{\ell} P_\ell(\cos\psi')$$

- We actually learned this in Math Methods, when we discussed the generating function for Legendre polynomials

$$G(x,h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

- So as long as r^2 is larger than the largest value of $r'^2 - 2rr'\cos\psi'$, the potential is:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int dq(\vec{r}\,') \, (r')^{\ell} P_{\ell}(\cos\psi')$$

- In other words: If we're far enough away from the charge, the complicated integral for V can be written as a series of progressively smaller terms.

- This is the "MULTIPOLE EXPANSION." The terms in the expansion are called "MULTIPOLES"

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int dq(\vec{r}\,') + \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int dq(\vec{r}\,') \, r'\cos\psi'$$

$$+ \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \int dq(\vec{r}\,') \, (r')^2 \times \left(\frac{3}{2}\cos^2\psi' - \frac{1}{2} \right) + \dots$$

- Start calculating terms, until you feel like the contribution to V from the next term is too small to care about. Congratulations – you have an approximation for the potential!

$$V(\vec{r}) = \overbrace{\underbrace{V_{mon}(\vec{r}) + V_{dip}(\vec{r})}_{\text{Keep just these? APPROXIMATE.}} + V_{quad}(\vec{r}) + \dots}^{\text{Keep all terms? EXACT.}}$$

- Notice that individual terms in the multipole expansion depend on r, r', & $\cos\psi'$, but <u>not</u> in the combination $|\vec{r} - \vec{r}\,'|$. So the multipole expansion depends on our choice of origin.

- The 'monopole' term ($\ell = 0$) just tells us how much the distribution of charge looks like a point charge <u>at the origin</u>.

Total charge

$$V_{\text{Mono}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int dq(\vec{r}\,') = \frac{q_{TOT}}{4\pi\varepsilon_0 r}$$

Pot. @ point $\vec{r}$ distance r from origin.

- The 'dipole' term ($\ell = 1$) begins to discern some details of how the charge is spread out.

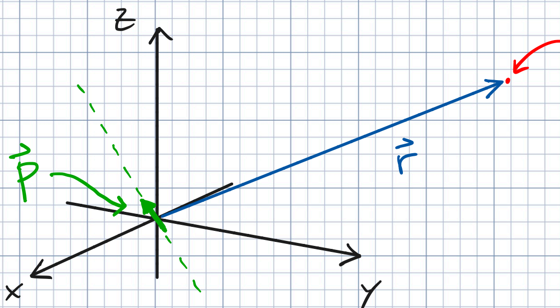
RECALL: $\hat{r}\cdot\hat{r}\,' = \cos\psi'$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int dq(\vec{r}\,')\,\underbrace{r'\cos\psi'}_{r'\hat{r}\,'\cdot\hat{r}\,=\,\vec{r}\,'\cdot\hat{r}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat{r}\cdot\Big(\underbrace{\int dq(\vec{r}\,')\,\vec{r}\,'}_{\text{'DIPOLE MOMENT' } \vec{p}}\Big)$$

Sum up all the charge, w/ each bit weighted by its position vector.

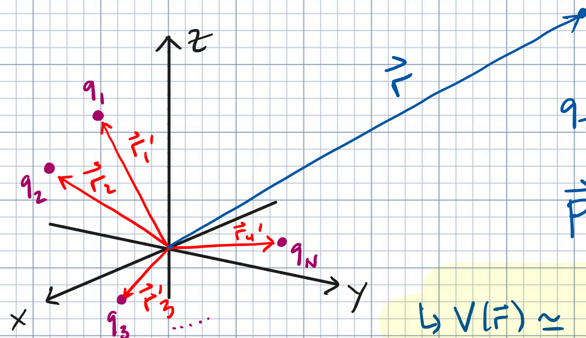
$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$



Next level of detail beyond pt. charge: the distribution 'looks' like a dipole $\vec{p}$ @ origin.

- Let's look @ some examples.

- First, suppose we have a collection of N pt. charges. We'll call them q_i, w/ $i = 1, \dots, N$. Now pick an origin, so position of q_i is $\vec{r}_i{}'$.



$$q_{TOT} = \sum_{i=1}^{N} q_i$$

$$\vec{p} = \sum_{i=1}^{N} q_i \vec{r}_i{}'$$

$$\hookrightarrow V(\vec{r}) \simeq \frac{\left(\sum_{i=1}^{N} q_i\right)}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0 r^2} \hat{r} \cdot \left(\sum_{i=1}^{N} q_i \vec{r}_i{}'\right)$$

- Of course, we know the exact result

$$V(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}_i{}'|}$$

But working out $|\vec{r} - \vec{r}_i{}'|^{-1}$ for all the charges (if N is large) might be too tedious if all we need is a 'good enough' approximation.

- How'd I get $\vec{p}$? Well, we said $\vec{p}$ is just the sum of all the charges w/ each one weighted (multiplied by) its position vector. So that's what I did. But if you want an integral:

$$\rho(\vec{r}) = \sum_{i=1}^{N} q_i \delta^3(\vec{r} - \vec{r}_i{}')$$ $\leftarrow$ $\rho = q\,\delta^3(\vec{r} - \vec{r}\,')$ for a point charge

$$\vec{p} = \int d\tau' \rho(\vec{r}\,') \vec{r}\,' = \sum_{i=1}^{N} q_i \int d\tau' \vec{r}\,' \delta^3(\vec{r} - \vec{r}_i{}') = \sum_{i=1}^{N} q_i \vec{r}_i{}'$$

- Suppose I have two charges: $q_1 = Q$ @ $\vec{r}_1{}' = L\hat{z}$ and $q_2 = 3Q$ @ $\vec{r}_2{}' = \frac{L}{2}\hat{x}$.

$$q_{TOT} = Q + 3Q = 4Q$$

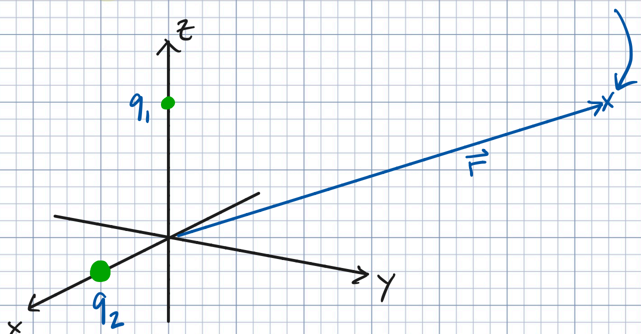
$$\vec{p} = q_1\vec{r}_1{}' + q_2\vec{r}_2{}' = QL\hat{z} + \frac{3}{2}QL\hat{x}$$

$\hat{x}\cdot\hat{r} = \sin\theta\cos\phi$

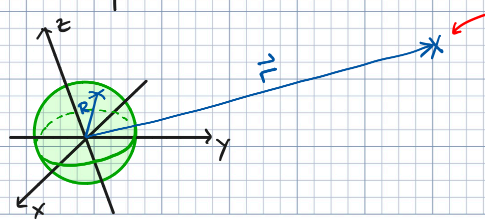
$\hat{z}\cdot\hat{r} = \cos\theta$

$$\hookrightarrow V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\frac{4Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{3}{2}QL\hat{x} + QL\hat{z}\right)\cdot\hat{r}}{r^2}$$

$$V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\left(\frac{4Q}{r} + \frac{\frac{3}{2}QL\sin\theta\cos\phi + QL\cos\theta}{r^2}\right)$$



- Second, let's consider a sphere of radius R w/ a surface charge density. It has azimuthal symmetry, w/ a σ that depends only on the polar angle θ. We'll call the axis of symmetry the z-axis, & put our origin @ the center of the sphere.



What's the approximate potential here, where $r \gg R$?

- The monopole moment is just q_{TOT}:

$$q_{TOT} = \int da' \sigma(\theta') = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' R^2 \sin\theta' \sigma(\theta')$$

$$= 2\pi R^2 \int_0^{\pi} d\theta' \sin\theta' \sigma(\theta')$$

w/out more info about $\sigma(\theta)$, this is all I can say about q_{TOT}.

- How about the dipole moment? With our origin @ the center of the sphere, a point on the surface has

$$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$= R\sin\theta'\cos\phi'\hat{x} + R\sin\theta'\sin\phi'\hat{y} + R\cos\theta'\hat{z}$$

$$\vec{p} = \int dq(\vec{r}')\,\vec{r}' = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' R^2 \sin\theta' \sigma(\theta') \times \left(R\sin\theta'\cos\phi'\hat{x} + R\sin\theta'\sin\phi'\hat{y} + R\cos\theta'\hat{z}\right)$$

$$= 0\,\hat{x} + 0\,\hat{y} + 2\pi R^3 \left(\int_0^{\pi} d\theta' \sin\theta' \cos\theta' \sigma(\theta')\right)\hat{z}$$

b/c $\int_0^{2\pi} d\phi' \cos\phi' = \int_0^{2\pi} d\phi' \sin\phi' = 0$

$$\hookrightarrow V(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{q_{TOT}}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2}\right)$$

For $r \gg R$!

w/ $q_{TOT} = 2\pi R^2 \int_0^{\pi} d\theta' \sin\theta' \sigma(\theta')$

$$\vec{p} = 2\pi R^3 \hat{z} \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \sigma(\theta')$$

Remember: I said there was an axis of symmetry, which we called $\hat{z}$!

- So what if we had $\sigma(\theta) = \sigma_0 + \sigma_1 \cos^3\theta$?

$$q_{TOT} = 2\pi R^2 \int_0^\pi d\theta' \sin\theta' \left(\sigma_0 + \sigma_1 \cos^3\theta'\right)$$

$$= 2\pi R^2 \times \left(2\sigma_0 + 0\cdot\sigma_1\right) = 4\pi R^2 \sigma_0$$

$$\vec{p} = 2\pi R^3 \hat{z} \times \int_0^\pi d\theta' \sin\theta' \cos\theta' \times \left(\sigma_0 + \sigma_1 \cos^3\theta'\right)$$

$$= 2\pi R^3 \hat{z} \times \left(0\,\sigma_0 + \frac{2}{5}\sigma_1\right)$$

$$= \frac{4}{5}\pi R^3 \sigma_1 \hat{z}$$

$$\hookrightarrow V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0} \times \left(\frac{4\pi R^2 \sigma_0}{r} + \frac{\frac{4}{5}\pi R^3 \sigma_1 \overbrace{\hat{z}\cdot\hat{r}}^{\cos\theta}}{r^2} \right)$$

- In this example, the constant part of $\sigma(\theta)$ contributes $4\pi R^2 \sigma_0$ to q_{TOT}, but nothing to $\vec{p}$. The $\sigma_1 \cos^3\theta$ is odd across the equator ($\theta = \pi/2$), so its contribution to $q_{TOT} = 0$ but it gives a dipole moment $(4\pi R^3 \sigma_1 / 5)\hat{z}$. There is no quadrupole moment ($\ell = 2$) but there <u>is</u> an octupole moment (the $\ell = 3$ term).